An imaginary temperature far away from a stationary spinning star

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By what appears to be a natural extension a temperature is assigned to the vacuum spacetime outside an isolated rigidly rotating star in an asymptotically flat stationary axisymmetric spacetime. This temperature becomes imaginary far away from the axis of the star turning the equation for the concentration of test particles in the few particles limit in this region into a Schrödinger type equation. A distance from the axis where the temperature gives Planck's constant is crudely estimated for various gravitating objects assuming linear and quadratic temperature dependence of the diffusion coefficient to argue for a gravitational origin of basic quantum mechanics. For a uniform density classical object having the mass and angular momentum comparable to that of a neutron and assuming quadratic temperature (magnitude) dependence of the diffusion coefficient with the proportionality constant $b k_B$ and with mobility $b = 2a/m$ (m mass of the test particle) we get Planck's constant roughly at 0.2 Bohr radius on the equatorial plane. In view of the crudeness of the various estimations involved it must be a remarkable coincidence and it calls for detail analysis using expertise from several branches of mathematical physics.

Keywords: Gravity, Quantum mechanics, Imaginary temperature, Rotating star.

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Introduction

We begin with a stationary axisymmetric perfect fluid solution modeling a rigidly rotating star surrounded by asymptotically flat vacuum. It has a metric of the form

$$g = g_{ab} dx^a dx^b = -V dt^2 + 2W dt d\varphi + X d\varphi^2 + \bar{g} \quad (1)$$

where $g = e^{2\mu}(d\rho^2 + Z dz^2)$, $\rho = \sqrt{VX + W^2}$, and V, W, X, μ, Z are functions of ρ and z . $X = 0$ on the axis and $X > 0$ elsewhere. Angular velocity of “the dragging of inertial frame” is $\omega = -W/X$. This means test particles with “zero angular momentum” move along trajectories whose angular velocity relative to a stationary observer at infinity is $\omega = d\varphi/dt$. Such an observer has 4-velocity $\frac{\partial}{\partial t}$ in his proper frame. ω is not the angular velocity of the fluid measured by the same observer. Energy-momentum tensor of perfect fluid is $T_{ab} = (\epsilon + p)u_a u_b + p g_{ab}$ where fluid's (normalized) 4-velocity is $u^a \frac{\partial}{\partial x^a} = \tau K^a \frac{\partial}{\partial x^a}$. It is known (see Lindblom [1]) that in thermal equilibrium the fluid moves rigidly, the temperature τ is constant along an integral curve

of u^a , and that K^a is a Killing vector field. τ is the temperature in the rest frame of the fluid. In thermal equilibrium the shear and expansion of the integral curves of u^a vanishes. If we write

$$K^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \quad (2)$$

then rigid rotation corresponds to a constant Ω inside the star [1]. The Killing vector K^a then extends outside the star to the whole of the stationary axisymmetric vacuum exterior. Inside the star τ^{-1} is the redshift factor so that

$$\tau = + \left(V - 2W\Omega - X\Omega^2 \right)^{-1/2} \quad (3)$$

We put the + sign to alert the reader for future discussion. The above equation (modulo a constant fixing the unit of the temperature) is a generalization of Tolman's equation $\tau = \sqrt{-g^{00}}$ for a fluid at rest [2], Thorne [3]. In the exterior we shall insist on the definition of τ using Eq.(3) with the same constant Ω . We may also include an extra factor k in the RHS and speculate on k changing with distance very slowly so that k can be considered constant locally. We shall use geometrized units $G = c = 1$. Since τ defined as a ratio of two 4-vectors inside the star is a dimensionless quantity, the relation $\epsilon + p = \tau s + \mu n$ (Eqs. 7a,7b in [1]) implies that entropy density s has the unit cm^{-2} . Eq.(2) gives $K^a K_a = g \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \varphi} \right) = -V + 2W\Omega + X\Omega^2$. In the fluid this becomes $K^a K_a = -\tau^{-2} \Leftrightarrow u^a u_a = -1$. If the fluid extends to infinity asymptotic flatness condition that $V \rightarrow 1, W \rightarrow 0$ and $X = O(\rho^2)$ gives $\Omega = 0$. Ω is called the angular velocity of the fluid element measured by a stationary observer in the asymptotically flat region in his rest frame.

Physically the temperature is not defined outside the star because u^a is not defined in the exterior. However in the exterior we can call τ defined by Eq.(3) a temperature or to be careful g-temperature of the vacuum. Here g stands for the gravity. g-temperature becomes imaginary if K^a becomes spacelike. Although it is defined far away from the axis of a stationary spinning star by a deeper analysis g-temperature could possibly be associated with any physical angular momentum carried locally in a much more general gravitational field. g-temperature may not be a mere mathematical construction. In the absence of the star the exterior geometry analytically extended inward does not close up the hole left without forming singularities or creating another end. g-

temperature is some measure of these distortions. For simplicity in this paper we assume that our spacetime has no ergoregions. For any constant β , $\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \varphi}$ is a Killing vector field of the spacetime metric of Eq.(1). It is only that in the exterior we choose a β which has a meaning on the surface of the star. Since we are overlooking how the star and the exterior came to be attached, we are overlooking the history of this vector field. But that should not belittle the fact that this vector field is special. Wald [6], Jacobson [7, 8] and several other authors demonstrated connections between Einstein equations and thermodynamic considerations. In defining g-temperature we are not following their path but because of the involvement of the Killing vector field we guess that there is some relation which remains to be explored. Before proceeding we note that in the static case Eq.(3) becomes $\tau = V^{-1/2}$. For a Schwarzschild black hole $V = 0$ on the event horizon. Thus τ is not the Bekenstein temperature [4] of a black hole surface unless the latter temperature is also defined in a limiting sense off the black hole surface. The Bekenstein temperature was defined using thermodynamic analogy that the area of a black hole is non-decreasing like entropy. It was further justified by Hawking [5] using quantum field theoretic arguments. We shall not consider black holes. Except for the occasional mentions of the universe, our spacetime is globally regular having space topology $\mathbb{R}^3$.

Planck's constant from g-temperature

Imaginary temperature would create a Schrödinger type equation for a test particle. We suppose that we are in the tail end far from the singularity in Eq.(3) so that we can consider locally the temperature to be constant. Let us imagine a test particle in the exterior vacuum at a place with imaginary g-temperature like a suspended molecule undergoing Brownian motion in a fluid in the limiting theory that the concentration of the suspended particles is small. If we assume roughly that the g-temperature is constant, Einstein relation for the diffusion coefficient D and mobility b gives $D = b\tau$. We suppose that Einstein relation also holds for imaginary temperature. Assuming that the temperature is measured in kelvin we take Einstein relation in the form $D = k_B b\tau$ where k_B is the Boltzmann constant. Such a simple model however does not give a comparable value of the Planck's constant for the Schrödinger type equation unless we manipulate the factor k we want to put in the Tolman-Thorne equation Eq.(3). An effect of a variable k can also be produced by not assuming a linear dependence of the diffusion coefficient on temperature which is used when temperature is constant. Thermal diffusion needs temperature gradient. We shall take the diffusion coefficient to be $D = i k_B b |\tau|^{1+\alpha}$.

Assuming b to be real we find that D is imaginary. We write $D = i\hat{D}$. Then in a geodesic normal coordinate the concentration $C(x, t)$ about a point satisfies the equation

$$\frac{\partial C}{\partial t} = \hat{D}i_{\Delta_{\hat{g}}}C - v \cdot \nabla C + \hat{D}\Gamma O(x^2)C \quad (4)$$

where Γ depends on the curvature components of the Riemannian 3-metric $\hat{g}$ at the origin of the coordinate system. Since inside the star τ was the temperature in the rest frame of the fluid and since in the exterior we have only gravitational field we assume that the drift $v = 0$. Then Eq.(4) gives

$$i\frac{\partial C}{\partial t} = -\hat{D}\nabla_{\hat{g}}C - \hat{D}\Gamma O(x^2)C \quad (5)$$

This is a Schrödinger type equation for complex C when $\hat{D} > 0$ or its complex conjugate C^* when $\hat{D} < 0$. Since we are in a stationary spacetime our direction of time corresponds to the sense of spin (rotation) but otherwise unimportant. Also in a more rigorous treatment one would possibly want to consider a norm devised from the symplectic inner product and compare $||C||^2$ with the probability density function of Brownian motion. But this and other rigorous investigations we shall leave for interested readers. My aim is to draw attention to some qualitative results. One can later try to make a better sense of mobility with a better model of Brownian motion. We find an estimate for the mobility b using Newtonian mechanics. Let the mass of the star be M and the asymptotically defined angular momentum per unit mass be a . Since gravitational force is the only force involved assuming Pascal's law we equate the speed ωr with $|b|$ times the gravitational force mM/r^2 and use $\omega = -W/X \approx 2Ma/r^3$ for large r . We find that heuristically for a test particle of mass m , $|b| = 2|a|/m$. Since for large r , $\tau = -i(|\Omega|r)^{-1} + O(r^{-2})$, taking a and Ω to be positive we get

$$\hat{D} \approx -\frac{2ak_B}{m(\Omega r)^{1+\alpha}} \quad (6)$$

We take $\hat{D} < 0$. Usually the signs of the mobilities are taken to be positive. For electrons and holes in the study of the p-n junction charges change the signs of the diffusion coefficients. In

our case b should be negative because drag force is directed opposite to the velocity. Equating $\hat{D}$ to $-\hbar_g/(2m)$ we then get an analogue of the reduced Planck's constant for our imaginary g-temperature:

$$\hbar_g \approx \frac{4ak_B}{(\Omega r)^{1+\alpha}} \quad (7)$$

For numerical estimation we remove a using the radius of the star. Let R be the average radius of the star. Angular momentum of the star is $J = Ma$. Using $J = (2/5)MR^2\Omega$ we get $a = (2/5)R^2\Omega$. Here 2/5 comes because we used the moment of inertia of a Maclaurin spheroid. It is not important at the level of our accuracy. Thus Eq. (7) gives

$$\hbar_g \approx \frac{1.6R^2k_B}{\Omega^\alpha r^{1+\alpha}} \quad (8)$$

Let ϵ_{ave} be the average density. Using $\Omega^2 \approx (4\pi/3)\epsilon_{\text{ave}}$ (which is derived by equating the value of the centrifugal force on a test particle and that of the Newtonian gravitational force on it) $\approx M/R^3$, we get

$$\hbar_g \approx \frac{1.6R^2k_B}{(4\pi/3)\epsilon_{\text{ave}} r^{1+\alpha}} \approx \frac{1.6R^{2+1.5\alpha}k_B}{M^{\alpha/2} r^{1+\alpha}} \quad (9)$$

Let $\alpha = 1$. For $R = R_\odot = 6.96 \times 10^{10}\text{cm}$ and $\epsilon_{\text{ave}} = \epsilon_\odot = 1.05 \times 10^{-28}\text{cm}^{-2}$ we get $\hbar_g \approx \hbar$ at $r \approx 10^{18}\text{cm}$. We recall that in geometrized units we have $1\text{ s} = 3 \times 10^{10}\text{cm}$, $1\text{ gm} = 0.7425 \times 10^{-28}\text{cm}$, $1\text{ erg} = 0.8264 \times 10^{-41}\text{cm}$ and $\hbar = 2.61 \times 10^{-66}\text{cm}^2$. Here r is the distance from the axis of symmetry. On the axis g-temperature never becomes imaginary because on the axis $X = 0$. Crudely speaking imaginary temperature comes out at a distance of $r_i \approx \Omega^{-1}$. So the imaginary temperature comes out at a distance of $r_i \approx \epsilon_{\text{ave}}^{-1/2} \approx 10^{14}\text{cm}$ from the axis. For a rotating stationary star the points on a $t = \text{constant}$ hypersurface where the Killing vector field K^a becomes null is a surface. Let us denote the minimum ρ -value on the intersection of this surface

with the equatorial plane by ρ_i . In case this minimum value occurs at a point where the asymptotic coordinate system is a valid approximation we denote the corresponding value by r_i .

After finding that $\hbar_g$ does not match $\hbar$ in isolated astrophysical objects at meaningful distances we consider modeling elementary particles with gravitation. Here we get a surprise. We assume that the Newtonian approximations crudely apply to a neutron and take R = neutron radius = 1.2×10^{-13} cm and M = neutron mass = 1.25×10^{-52} cm. First we estimate the distance at which g-temperature becomes imaginary. If we continue to believe that this distance is of the order of Ω^{-1} (in geometrized unit) and angular velocity (angular frequency) of the neutron is related to its spin $\frac{1}{2}$ by $\Omega = 2\hbar M^{-1}R^{-2}$ then this distance is approximately 3.4×10^{-13} cm $\approx 0.6 \times 10^{-4}$ Bohr radius. We estimate $\hbar_g$ for the case $\alpha = 1$. Eq.(9) gives $\hbar_g \approx \hbar$ at $r \approx 10^{-9}$ cm ≈ 0.2 Bohr radius. This does not seem bad because of the crudeness in the estimating process. A question arises about the level surfaces of the imaginary temperature. They are not spherical in axisymmetric rotating objects. Does this fact have any connection with the so-called spatial quantization in basic quantum mechanics?

Discussion

One may get a shock that $\hbar_g$, which should be the reduced Planck's constant $\hbar$, is vanishing far away from the star. This happens because the g-temperature decays at large r . It could be related to the quantum mechanical fact that in the limit $\hbar \rightarrow 0$ we get Newtonian mechanics. The important point, however, is that $\hbar_g$ is vanishing for only one star and if there is a gravitating particle like neutron anywhere in the universe, there will be a halo of imaginary g-temperature and $\hbar_g \approx \hbar$ at appropriate distance from its axis. In this average sense, as far as quantum particles are concerned, $\hbar_g$ will be almost the same near the particles even when the particles are at astronomical distance from us. Another shock results when we see that an isolated astronomical object would produce a Planck type constant at some distance from it. Fortunately this distance is enormous. Its enormity reminds the well-known problem associated with large powers of ten separating the cosmic scale from Planck scale. For an astronomical object one may wish to push this distance towards the boundary of the visible universe. Taking $\alpha = 0$, that is, $D = k_B b \tau$ we find $\hbar_g \approx 1.6R^2 k_B / r$. This gives $\hbar_g \approx \hbar$ at about $r \approx 10^{22}$ cm for a star. Let us forget the issue of the Killing vector field and matching the vacuum at the surface of the star for a moment. One feels that imaginary g-temperature will in general also appear at some places outside a non-perfect fluid, non-stationary or non-rigidly rotating star in a realistic universe. In any case we possibly cannot

produce g-quantum effects for normal astrophysical objects at a manageable distance. Since the axes of the stars may have different orientations and the universe is expanding, g-temperature of far way stars may not add up. Our setting of a single star in an asymptotically flat space is not satisfactory in the universe with a cosmic horizon. On the onset of the imaginary temperature according to Eq.(3), τ is unbounded. The resolution of this problem may require some history of the formation of the star and the exterior as well as a model of the entropy-energy relation. If we consider the star to spin up from rest always rotating rigidly then as angular velocity increases the region of imaginary temperature seems to expand inwards from infinity. This region of imaginary temperature is bounded in the inner side by the surface where the Killing vector field becomes null. Thus maximum angular velocity may correspond to some sort of maximum entropy that can be associated with some part of the exterior. On the other hand uniqueness or rigidity type results indicate some upper bound for the energy.

Open questions

One would like to look for highly compact rapidly rotating axisymmetric gravitating objects not made of perfect fluid in an asymptotically flat empty space or space with an electromagnetic field. Previous example of neutron suggests modeling the interior with coupled Einstein-Yang-Mills with or without other fields. The observation of Bartnik and McKinnon [10] that for their particles mass-to-radius ratio is approximately 1 and the “regularizing effect” of gravity discussed by Finster, Smoller and Yau [11] for their static solutions of Einstein-Dirac-Maxwell equations also suggest for such an undertaking. Rotating stationary axisymmetric solutions are rare or nonexistent (Biz and Radu [12]). There are rotating boson stars found by Yoshida and Eriguchi [13]. For a globally regular solution without the attached exterior vacuum, the Killing vector K^a (or its appropriate analogue) may not become null unless there is an ergoregion. In such a situation without the ergoregion g-temperature may not become imaginary. On the other hand many solutions are numerically constructed so that error in matching the exterior may result or exclude imaginary g-temperature. We leave these investigations for interested readers. Although at present there is no known solution matching rigorously vacuum exterior to a spinning star, Heilig [14] has shown that they exist for Einstein’s equations of Ehler’s frame theory in the neighborhood of a Newtonian star and small angular velocity. See also MacCallum, Mars, Vera [15]. There are however numerically matched exact or numerical solutions. Pachon, Rueda and Sanabria-Gomez [16] found exact vacuum or electromagnetic exteriors matched with numerical interiors. Pappas and Apostolatos [17] found exterior solutions matched numerically

with a neutron star. The imaginary temperature we are getting is because of the rotation. In general a magnetic field outside a spinning star is a very natural phenomenon. Although the Killing vector field producing this imaginary temperature has associated with it a magnetic field or generally an electromagnetic field (Wald [18]) in the vacuum, one should also consider exteriors made up of solutions of Einstein-Maxwell coupled equations. Only physical example of imaginary temperature we found in the literature is associated with antiferromagnetism. At this point one recalls the two roots of Eq.(3). For the Schwarzschild solution the possible negative square root of Eq.(3) occurs on the image obtained by doubling across the horizon. For a single rotating star in the region where imaginary temperature comes about, two roots correspond to C and C^* at the same point. Orientations of stars in a group of stars may provide an analogue of spin-glass type models. We glossed over the involvement of Planck's constant $\hbar$ in the surface temperature of the star in choosing the expression for the Einstein relation. The surface temperature of a star comes about from the choice of the equation of state of the fluid. The equation of state is constructed assuming quantum related matter. Whether the star radiates as a black body or it is a silver sphere, in general, the expression of the surface temperature would involve the Planck's constant. Physically surface temperature of the star is related to the luminosity. Some equation (for example Stefan-Boltzmann law in the case of a black body) relates the luminosity with the surface temperature. Clearly our analysis can be improved here. Involvement of Planck's constant in the coupling constants, for example of Einstein-Yang-Mills equations, may create new problems because to be consistent we may have to replace the constant $\hbar$ by variable $\hbar_g$.

Conclusion

We did not try to derive Einstein equations from thermodynamic laws. We consider Einstein equations as a fundamental law of physics creating the inertial mass and the energy-momentum tensor of various types of fields. Thermodynamic laws are of mathematical or statistical origin. Similarly, except for Planck's constant, quantum physics is of mathematical origin associated, roughly speaking, with certain type of differential equations having variational formulation, and formulation in terms of operators having discrete spectra. Physicists invented quantum mechanics but now people can quantize finance. What is meant by a unified theory when the aim is to unify different subjects on the same footing? We are trying to see everything in Einstein equations. In Einstein equations, the metric and its derivatives up to second order gives the energy-momentum tensor. The only task remains is to extract the matter fields from this

energy-momentum tensor. In the present paper we found a Schrödinger type equation with a suitable Planck type “constant.”

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